

Consecutive square-free numbers of a special form

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Abstract

In the present paper we give an asymptotic formula for the number of pairs of consecutive square-free numbers of a special form.

Keywords: Square-free numbers, Exponential sums.

1 Notations.

Let X be a sufficiently large positive number. By ε we denote an arbitrary small positive number, not necessarily the same in different occurrences. We denote by $\mu(n)$ the Möbius function and by $\tau(n)$ the number of positive divisors of n . As usual $[t]$ and $\{t\}$ denote the integer part, respectively, the fractional part of t . Let $||t||$ be the distance from t to the nearest integer. Instead of $m \equiv n \pmod{k}$ we write for simplicity $m \equiv n(k)$. As usual $e(t) = \exp(2\pi it)$. For positive A and B we write $A \asymp B$ instead of $B \ll A \ll B$. Let c be a real constant such that $1 < c < 3/2$.

Denote

$$z = X^{\frac{3-2c}{2}}; \quad (1)$$

$$\gamma = 1/c; \quad (2)$$

$$\psi(t) = \{t\} - 1/2; \quad (3)$$

$$\sigma = \prod_p \left(1 - \frac{2}{p^2}\right). \quad (4)$$

2 Introduction and statement of the result.

The problem for the consecutive square-free numbers arises in 1932 when Carlitz [4]

proved that

$$\sum_{n \leq X} \mu^2(n) \mu^2(n+1) = \sigma X + \mathcal{O}(X^{\frac{2}{3}+\epsilon}), \quad (5)$$

where σ is denoted by (4). Formula (5) was subsequently improved by Heath-Brown [5] and by Reuss [6].

On the other hand in 2008 Cao and Zhai [3] proved that for any fixed $1 < c < 149/87$ the asymptotic formula

$$\sum_{n \leq X} \mu^2([n^c]) = \frac{6}{\pi^2} X + \mathcal{O}(X^{1-\epsilon}) \quad (6)$$

holds. Their earlier result [2] covers the narrower range $1 < c < 61/36$.

Define

$$S_c(X) = \sum_{X/2 < n \leq X} \mu^2([n^c]) \mu^2([n^c] + 1). \quad (7)$$

We couple the theorems (5) and (6) by proving

Theorem 1. *Let $1 < c < 3/2$. Then the asymptotic formula*

$$S_c(X) = \frac{1}{2} \sigma X + \mathcal{O}\left(X^{\frac{2c+1}{4}+\epsilon}\right),$$

holds. Here σ is defined by (4).

3 Lemmas.

Lemma 1. *Let $|f^{(m)}(x)| \asymp Y X^{1-m}$ for $1 < X < x \leq 2X$ and $m = 1, 2, \dots$*

Then

$$\sum_{X < n \leq 2X} e(f(n)) \ll Y^\kappa X^\lambda + Y^{-1},$$

where (κ, λ) is any exponent pair.

Proof. See [1]. □

Lemma 2. *Let $n \in \mathbb{N}$. Then*

$$\tau(n) \ll n^\epsilon.$$

4 Proof of the Theorem.

We use (7) and the well-known identity $\mu^2(n) = \sum_{d^2|n} \mu(d)$ to write

$$\begin{aligned}
S_c(X) &= \sum_{X/2 < n \leq X} \sum_{d^2 | [n^c]} \mu(d) \sum_{t^2 | [n^c] + 1} \mu(t) \\
&= \sum_{\substack{d, t \\ (d, t) = 1}} \mu(d) \mu(t) \sum_{\substack{X/2 < n \leq X \\ [n^c] \equiv 0 \pmod{d^2} \\ [n^c] + 1 \equiv 0 \pmod{t^2}}} 1 \\
&= \sum_{\substack{d, t \\ (d, t) = 1}} \mu(d) \mu(t) \sum_{\substack{((X/2)^c - 1)d^{-2} < k \leq X^c d^{-2} \\ kd^2 + 1 \equiv 0 \pmod{t^2}}} \sum_{\substack{X/2 < n \leq X \\ [n^c] = kd^2}} 1 \\
&= \sum_{\substack{d, t \\ (d, t) = 1}} \mu(d) \mu(t) \sum_{\substack{((X/2)^c - 1)d^{-2} < k \leq X^c d^{-2} \\ kd^2 + 1 \equiv 0 \pmod{t^2}}} \sum_{\substack{X/2 < n \leq X \\ kd^2 \leq n^c < kd^2 + 1}} 1 \\
&= \sum_{\substack{d, t \\ (d, t) = 1}} \mu(d) \mu(t) \sum_{\substack{(X/2)^c d^{-2} < k \leq X^c d^{-2} \\ kd^2 + 1 \equiv 0 \pmod{t^2}}} \sum_{(kd^2)^\gamma \leq n < (kd^2 + 1)^\gamma} 1 + \mathcal{O}(X^\varepsilon) \\
&= S_c^{(1)}(X) + S_c^{(2)}(X) + \mathcal{O}(X^\varepsilon), \tag{8}
\end{aligned}$$

where

$$S_c^{(1)}(X) = \sum_{\substack{dt \leq z \\ (d, t) = 1}} \mu(d) \mu(t) \sum_{\substack{(X/2)^c d^{-2} < k \leq X^c d^{-2} \\ kd^2 + 1 \equiv 0 \pmod{t^2}}} \sum_{(kd^2)^\gamma \leq n < (kd^2 + 1)^\gamma} 1, \tag{9}$$

$$S_c^{(2)}(X) = \sum_{\substack{dt > z \\ (d, t) = 1}} \mu(d) \mu(t) \sum_{\substack{(X/2)^c d^{-2} < k \leq X^c d^{-2} \\ kd^2 + 1 \equiv 0 \pmod{t^2}}} \sum_{(kd^2)^\gamma \leq n < (kd^2 + 1)^\gamma} 1. \tag{10}$$

Estimation of $S_c^{(1)}(X)$.

From (9) we have

$$\begin{aligned}
S_c^{(1)}(X) &= \sum_{\substack{dt \leq z \\ (d, t) = 1}} \mu(d) \mu(t) \sum_{\substack{(X/2)^c d^{-2} < k \leq X^c d^{-2} \\ kd^2 + 1 \equiv 0 \pmod{t^2}}} ([-(kd^2)^\gamma] - [-(kd^2 + 1)^\gamma]) \\
&= S_c^{(3)}(X) + S_c^{(4)}(X), \tag{11}
\end{aligned}$$

where

$$S_c^{(3)}(X) = \sum_{\substack{dt \leq z \\ (d, t) = 1}} \mu(d) \mu(t) \sum_{\substack{(X/2)^c d^{-2} < k \leq X^c d^{-2} \\ kd^2 + 1 \equiv 0 \pmod{t^2}}} \left((kd^2 + 1)^\gamma - (kd^2)^\gamma \right), \tag{12}$$

$$S_c^{(4)}(X) = \sum_{\substack{dt \leq z \\ (d, t) = 1}} \mu(d) \mu(t) \sum_{\substack{(X/2)^c d^{-2} < k \leq X^c d^{-2} \\ kd^2 + 1 \equiv 0 \pmod{t^2}}} \left(\psi(-(kd^2 + 1)^\gamma) - \psi(-(kd^2)^\gamma) \right). \tag{13}$$

First we shall estimate $S_c^{(3)}(X)$.

Using (12) and Abel's transformation we obtain

$$\begin{aligned}
S_c^{(3)}(X) &= \sum_{\substack{dt \leq z \\ (d,t)=1}} \mu(d)\mu(t) \sum_{\substack{(X/2)^c d^{-2} < k \leq X^c d^{-2} \\ kd^2+1 \equiv 0 \pmod{t^2}}} (kd^2)^\gamma \left(\gamma(kd^2)^{-1} + \mathcal{O}((kd^2)^{-2}) \right) \\
&= \gamma \sum_{\substack{dt \leq z \\ (d,t)=1}} \mu(d)\mu(t) d^{2(\gamma-1)} \sum_{\substack{(X/2)^c d^{-2} < k \leq X^c d^{-2} \\ kd^2+1 \equiv 0 \pmod{t^2}}} k^{\gamma-1} \\
&\quad + \mathcal{O} \left(\sum_{\substack{dt \leq z \\ (d,t)=1}} d^{2(\gamma-2)} \sum_{\substack{(X/2)^c d^{-2} < k \leq X^c d^{-2} \\ kd^2+1 \equiv 0 \pmod{t^2}}} k^{\gamma-2} \right) \\
&= \gamma \sum_{\substack{dt \leq z \\ (d,t)=1}} \mu(d)\mu(t) d^{2(\gamma-1)} \left[(X^c d^{-2})^{\gamma-1} \sum_{\substack{(X/2)^c d^{-2} < k \leq X^c d^{-2} \\ kd^2+1 \equiv 0 \pmod{t^2}}} 1 \right. \\
&\quad \left. - \int_{(X/2)^c d^{-2}}^{X^c d^{-2}} \left(\sum_{\substack{(X/2)^c d^{-2} < k \leq y \\ kd^2+1 \equiv 0 \pmod{t^2}}} 1 \right) (y^{\gamma-1})' dy \right] + \mathcal{O} \left(X^{1-2c} \sum_{\substack{dt \leq z \\ (d,t)=1}} \sum_{\substack{(X/2)^c d^{-2} < k \leq X^c d^{-2} \\ kd^2+1 \equiv 0 \pmod{t^2}}} 1 \right) \\
&= \gamma \sum_{\substack{dt \leq z \\ (d,t)=1}} \mu(d)\mu(t) d^{2(\gamma-1)} \left[X^{1-c} d^{2(1-\gamma)} \left(\frac{2^c-1}{2^c} \frac{X^c}{d^2 t^2} + \mathcal{O}(1) \right) \right. \\
&\quad \left. - \int_{(X/2)^c d^{-2}}^{X^c d^{-2}} \left(\frac{y - (X/2)^c d^{-2}}{t^2} + \mathcal{O}(1) \right) (y^{\gamma-1})' dy \right] + \mathcal{O} \left(X^{1-c} \sum_{dt \leq z} \frac{1}{d^2 t^2} \right) \\
&= \frac{1}{2} X \sum_{\substack{dt \leq z \\ (d,t)=1}} \frac{\mu(d)\mu(t)}{d^2 t^2} + \mathcal{O} \left(X^{1-c} \sum_{dt \leq z} 1 \right) + \mathcal{O}(1) \\
&= \frac{1}{2} X \sum_{\substack{d,t=1 \\ (d,t)=1}} \frac{\mu(d)\mu(t)}{d^2 t^2} - \frac{1}{2} X \sum_{\substack{dt > z \\ (d,t)=1}} \frac{\mu(d)\mu(t)}{d^2 t^2} + \mathcal{O} \left(X^{1-c} \sum_{dt \leq z} 1 \right) + \mathcal{O}(1). \quad (14)
\end{aligned}$$

It is well-known that

$$\sum_{\substack{d,t=1 \\ (d,t)=1}} \frac{\mu(d)\mu(t)}{d^2 t^2} = \prod_p \left(1 - \frac{2}{p^2} \right), \quad (15)$$

see ([7], Theorem 2.6.8).

On the other hand by Lemma 2

$$\sum_{\substack{dt > z \\ (d,t)=1}} \frac{\mu(d)\mu(t)}{d^2 t^2} \ll \sum_{dt > z} \frac{1}{d^2 t^2} = \sum_{n > z} \frac{\tau(n)}{n^2} \ll \sum_{n > z} \frac{1}{n^{2-\varepsilon}} \ll z^{\varepsilon-1}. \quad (16)$$

By the same way

$$\sum_{dt \leq z} 1 = \sum_{n \leq z} \tau(n) \ll z X^\varepsilon. \quad (17)$$

Bearing in mind (1), (4) and (14) – (17) we find

$$S_c^{(3)}(X) = \frac{1}{2} \sigma X + \mathcal{O}\left(X^{\frac{2c+1}{4} + \varepsilon}\right). \quad (18)$$

Now we shall estimate $S_c^{(4)}(X)$.

Replace

$$\Phi(k, d) = \psi(-(kd^2 + 1)^\gamma) - \psi(-(kd^2)^\gamma). \quad (19)$$

Let $M \geq 2$ is a real parameter, we shall choose latter depending on X, d and t . Using ([7], Lemma 5.2.2) we get

$$\Phi(k, d) = \frac{1}{2\pi i} \sum_{1 \leq |h| \leq M} \frac{\omega(k, d, h)}{h} + \mathcal{O}(\omega_1(k, d)) + \mathcal{O}(\omega_2(k, d)), \quad (20)$$

where

$$\omega(k, d, h) = e(-h(kd^2 + 1)^\gamma) - e(-h(kd^2)^\gamma), \quad (21)$$

$$\omega_1(k, d) = \min\left(1, \frac{1}{M|| - (kd^2 + 1)^\gamma ||}\right), \quad \omega_2(k, d) = \min\left(1, \frac{1}{M|| - (kd^2)^\gamma ||}\right). \quad (22)$$

From (13), (19), (20) and (22) it follows

$$S_c^{(4)}(X) = \frac{1}{2\pi i} \sum_{\substack{dt \leq z \\ (d, t)=1}} \mu(d)\mu(t) \sum_{1 \leq |h| \leq M} \frac{1}{h} \sum_{\substack{(X/2)^c d^{-2} < k \leq X^c d^{-2} \\ kd^2 + 1 \equiv 0 \pmod{t^2}}} \omega(k, d, h) + \Omega_1 + \Omega_2, \quad (23)$$

where Ω_1 and Ω_2 are the contributions of the remainder terms in (20).

Using ([7], Lemma 5.2.3) we obtain

$$\begin{aligned} \Omega_1, \Omega_2 &\ll \sum_{\substack{dt \leq z \\ (d, t)=1}} \sum_{\substack{(X/2)^c d^{-2} < k \leq X^c d^{-2} + 1/d^2 \\ kd^2 + 1 \equiv 0 \pmod{t^2}}} \min\left(1, \frac{1}{M|| (kd^2)^\gamma ||}\right) \\ &\ll \sum_{\substack{dt \leq z \\ (d, t)=1}} \sum_{\substack{(X/2)^c d^{-2} < k \leq X^c d^{-2} + 1/d^2 \\ kd^2 + 1 \equiv 0 \pmod{t^2}}} \sum_{h \in \mathbb{Z}} b_M(h) e(h(kd^2)^\gamma), \end{aligned} \quad (24)$$

where

$$b_M(h) \ll \begin{cases} \frac{\log M}{M} & \text{if } h \in \mathbb{Z}, \\ \frac{M}{h^2} & \text{if } h \in \mathbb{Z} \setminus \{0\}. \end{cases} \quad (25)$$

From (24) and (25) we find

$$\Omega_1, \Omega_2 \ll \sum_{dt \leq z} \frac{X^c}{d^2 t^2} \frac{\log M}{M} + \sum_{dt \leq z} \frac{\log M}{M} \sum_{1 \leq |h| \leq M} |H(t, h)| + \sum_{dt \leq z} M \sum_{|h| > M} \frac{|H(t, h)|}{h^2}, \quad (26)$$

where

$$H(t, h) = \sum_{((X/2)^c + 1)t^{-2} < l \leq (X^c + 2)t^{-2}} e(h(lt^2 - 1)^\gamma). \quad (27)$$

Denote

$$f(y) = h(yt^2 - 1)^\gamma.$$

We have

$$|f^{(m)}(y)| \asymp hX^{1-c}t^2(X^c t^{-2})^{1-m} \quad (28)$$

for

$$y \in \left(\frac{(X/2)^c + 1}{t^2}, \frac{X^c + 2}{t^2} \right].$$

Bearing in mind (27), (28) and Lemma 1 with $(\kappa, \lambda) = (\frac{1}{5}, \frac{1}{6})$ we get

$$H(t, h) \ll (hX^{1-c}t^2)^{\frac{1}{5}}(X^c t^{-2})^{\frac{1}{6}} + (hX^{1-c}t^2)^{-1}. \quad (29)$$

Taking $M = X^{\frac{2c-1}{3}}(dt)^{-\frac{2}{3}}$, by (1), (26), (29) and Lemma 2 we obtain

$$\begin{aligned} \Omega_1, \Omega_2 &\ll \sum_{dt \leq z} \frac{X^c}{d^2 t^2} \frac{\log M}{M} + \sum_{dt \leq z} \frac{\log M}{M} \sum_{h \leq M} (h^{\frac{1}{5}} t^{\frac{1}{15}} X^{\frac{6-c}{30}} + h^{-1} t^{-2} X^{c-1}) \\ &\quad + \sum_{dt \leq z} M \sum_{h > M} (h^{-\frac{9}{5}} t^{\frac{1}{15}} X^{\frac{6-c}{30}} + h^{-3} t^{-2} X^{c-1}) \\ &\ll X^{c+\varepsilon} \sum_{dt \leq z} M^{-1} (dt)^{-2} + X^{\frac{6-c}{30}+\varepsilon} \sum_{dt \leq z} M^{\frac{1}{5}} t^{\frac{1}{15}} + X^{c-1+\varepsilon} \sum_{dt \leq z} M^{-1} t^{-2} \\ &\ll X^{\frac{c+1}{3}+\varepsilon} \sum_{dt \leq z} (dt)^{-\frac{4}{3}} + X^{\frac{3c+4}{30}+\varepsilon} \sum_{dt \leq z} (dt)^{-\frac{1}{15}} + X^{\frac{c-2}{3}+\varepsilon} \sum_{dt \leq z} (dt)^{\frac{2}{3}} \\ &= X^{\frac{c+1}{3}+\varepsilon} \sum_{n \leq z} \tau(n) n^{-\frac{4}{3}} + X^{\frac{3c+4}{30}+\varepsilon} \sum_{n \leq z} \tau(n) n^{-\frac{1}{15}} + X^{\frac{c-2}{3}+\varepsilon} \sum_{n \leq z} \tau(n) n^{\frac{2}{3}} \\ &\ll X^{\frac{c+1}{3}+\varepsilon} \sum_{n \leq z} n^{-\frac{4}{3}} + X^{\frac{3c+4}{30}+\varepsilon} \sum_{n \leq z} n^{-\frac{1}{15}} + X^{\frac{c-2}{3}+\varepsilon} \sum_{n \leq z} n^{\frac{2}{3}} \\ &\ll X^{\frac{c+1}{3}+\varepsilon} z^{-\frac{1}{3}} + X^{\frac{3c+4}{30}+\varepsilon} z^{\frac{14}{15}} + X^{\frac{c-2}{3}+\varepsilon} z^{\frac{5}{3}} \\ &\ll X^{\frac{2c+1}{4}+\varepsilon}. \end{aligned} \quad (30)$$

From (23) and (30) it follows

$$S_c^{(4)}(X) \ll \sum_{dt \leq z} \sum_{h \leq M} \frac{1}{h} |S_c^{(5)}(X)| + X^{\frac{2c+1}{4}+\varepsilon}, \quad (31)$$

where

$$S_c^{(5)}(X) = \sum_{\substack{(X/2)^c d^{-2} < k \leq X^c d^{-2} \\ kd^2 + 1 \equiv 0 \pmod{t^2}}} \omega(k, d, h). \quad (32)$$

We have

$$\omega(k, d, h) = \theta_h(kd^2) e(-h(kd^2)^\gamma), \quad (33)$$

where

$$\theta_h(t) = e(h(t^\gamma - (t+1)^\gamma)) - 1. \quad (34)$$

Using (32), (33) and Abel's transformation we find

$$\begin{aligned} S_c^{(5)}(X) &= \sum_{\substack{(X/2)^c d^{-2} < k \leq X^c d^{-2} \\ kd^2 + 1 \equiv 0 \pmod{t^2}}} \theta_h(kd^2) e(-h(kd^2)^\gamma) \\ &= \theta_h(X^c) \sum_{\substack{(X/2)^c d^{-2} < k \leq X^c d^{-2} \\ kd^2 + 1 \equiv 0 \pmod{t^2}}} e(-h(kd^2)^\gamma) \\ &\quad - \int_{(X/2)^c d^{-2}}^{X^c d^{-2}} (\theta_h(yd^2))' \sum_{\substack{(X/2)^c d^{-2} < k \leq y \\ kd^2 + 1 \equiv 0 \pmod{t^2}}} e(-h(kd^2)^\gamma) dy \\ &\ll \left(|\theta_h(X^c)| + d^2 \int_{(X/2)^c d^{-2}}^{X^c d^{-2}} |\theta_h'(yd^2)| dy \right) \max_{y \in [(X/2)^c d^{-2}, X^c d^{-2}]} |S_c^{(6)}(X, y)|, \end{aligned} \quad (35)$$

where

$$S_c^{(6)}(X, y) = \sum_{\substack{(X/2)^c d^{-2} < k \leq y \\ kd^2 + 1 \equiv 0 \pmod{t^2}}} e(h(kd^2)^\gamma). \quad (36)$$

Consider $\theta_h(X^c)$. By (34) and the mean-value theorem we get

$$|\theta_h(X^c)| \leq 2 |\sin(\pi h((X^c)^\gamma - (X^c + 1)^\gamma))| \ll |h| |(X^c)^\gamma - (X^c + 1)^\gamma| \ll |h| X^{1-c}. \quad (37)$$

On the other hand

$$\theta_h'(t) = 2\pi i h \gamma (t^{\gamma-1} - (t+1)^{\gamma-1}) e(h(t^\gamma - (t+1)^\gamma)) \ll |h| t^{\gamma-2}$$

therefore

$$\theta_h'(t) \ll |h| X^{1-2c} \quad \text{for } t \in [(X/2)^c, X^c]. \quad (38)$$

Bearing in mind (35) – (38) we obtain

$$S_c^{(5)}(X) \ll |h| X^{1-c} \max_{y \in [(X/2)^c d^{-2}, X^c d^{-2}]} |S_c^{(6)}(X, y)|. \quad (39)$$

Thus from (31), (36) and (39) it follows

$$S_c^{(4)}(X) \ll X^{1-c} \sum_{dt \leq z} \sum_{h \leq M} \max_{y \in [(X/2)^c d^{-2}, X^c d^{-2}]} |S_c^{(6)}(X, y)| + X^{\frac{2c+1}{4}+\varepsilon}. \quad (40)$$

By (36) we have

$$S_c^{(6)}(X, y) = \sum_{((X/2)^c+1)t^{-2} < l \leq (yd^2+2)t^{-2}} e(h(lt^2 - 1)^\gamma)$$

and arguing as in (27) we find

$$\max_{y \in [(X/2)^c d^{-2}, X^c d^{-2}]} |S_c^{(6)}(X, y)| \ll (hX^{1-c}t^2)^{\frac{1}{5}}(X^c t^{-2})^{\frac{1}{6}} + (hX^{1-c}t^2)^{-1}. \quad (41)$$

Taking $M = X^{\frac{2c-1}{3}}(dt)^{-\frac{2}{3}}$, by (1), (17), (40), (41) and Lemma 2 we get

$$\begin{aligned} S_c^{(4)}(X) &\ll X^{\frac{36-31c}{30}} \sum_{dt \leq z} \sum_{h \leq M} h^{\frac{1}{5}} t^{\frac{1}{15}} + \sum_{dt \leq z} \sum_{h \leq M} h^{-1} t^{-2} + X^{\frac{2c+1}{4}+\varepsilon} \\ &\ll X^{\frac{36-31c}{30}} \sum_{dt \leq z} M^{\frac{6}{5}} t^{\frac{1}{15}} + X^\varepsilon \sum_{dt \leq z} 1 + X^{\frac{2c+1}{4}+\varepsilon} \\ &\ll X^{\frac{24-7c}{30}+\varepsilon} \sum_{dt \leq z} (dt)^{-\frac{23}{15}} + X^{\frac{2c+1}{4}+\varepsilon} \\ &= X^{\frac{24-7c}{30}+\varepsilon} \sum_{n \leq z} \tau(n) n^{-\frac{11}{15}} + X^{\frac{2c+1}{4}+\varepsilon} \\ &\ll X^{\frac{24-7c}{30}+\varepsilon} \sum_{n \leq z} n^{-\frac{11}{15}} + X^{\frac{2c+1}{4}+\varepsilon} \\ &\ll X^{\frac{24-7c}{30}+\varepsilon} z^{\frac{4}{15}} + X^{\frac{2c+1}{4}+\varepsilon} \\ &\ll X^{\frac{2c+1}{4}+\varepsilon}. \end{aligned} \quad (42)$$

Bearing in mind (11), (18) and (42) we obtain

$$S_c^{(1)}(X) = \frac{1}{2} \sigma X + \mathcal{O}\left(X^{\frac{2c+1}{4}+\varepsilon}\right). \quad (43)$$

Estimation of $S_c^{(2)}(\mathbf{X})$.

Using (10) we write

$$S_c^{(2)}(X) \ll (\log X)^2 \sum_{D \leq d < 2D} \sum_{T \leq t < 2T} \sum_{\substack{(X/2)^c d^{-2} < k \leq X^c d^{-2} \\ kd^2+1 \equiv 0 \pmod{t^2}}} \sum_{(kd^2)^\gamma \leq n < (kd^2+1)^\gamma} 1, \quad (44)$$

where

$$\frac{1}{2} \leq D, T \leq \sqrt{X^c+1}, \quad DT \geq \frac{z}{4}. \quad (45)$$

On the one hand (44) and Lemma 2 give us

$$\begin{aligned}
S_c^{(2)}(X) &\ll (\log X)^2 \sum_{T \leq t < 2T} \sum_{l \leq (X^c+1)T^{-2}} \sum_{D \leq d < 2D} \sum_{\substack{(X/2)^c d^{-2} < k \leq X^c d^{-2} \\ kd^2 = lt^2 - 1}} \sum_{(kd^2)^\gamma \leq n < (kd^2+1)^\gamma} 1 \\
&\ll (\log X)^2 X^{1-c} \sum_{T \leq t < 2T} \sum_{l \leq (X^c+1)T^{-2}} \tau(lt^2 - 1) \\
&\ll X^{1-c+\varepsilon} \sum_{T \leq t < 2T} \sum_{l \leq (X^c+1)T^{-2}} 1 \\
&\ll X^{1+\varepsilon} T^{-1}.
\end{aligned} \tag{46}$$

On the other hand (44) and Lemma 2 imply

$$\begin{aligned}
S_c^{(2)}(X) &\ll (\log X)^2 \sum_{D \leq d < 2D} \sum_{k \leq X^c D^{-2}} \sum_{l \leq (X^c+1)T^{-2}} \sum_{\substack{T \leq t < 2T \\ kd^2+1=lt^2}} \sum_{(kd^2)^\gamma \leq n < (kd^2+1)^\gamma} 1 \\
&\ll (\log X)^2 X^{1-c} \sum_{D \leq d < 2D} \sum_{k \leq X^c D^{-2}} \tau(kd^2 + 1) \\
&\ll X^{1-c+\varepsilon} \sum_{D \leq d < 2D} \sum_{k \leq X^c D^{-2}} 1 \\
&\ll X^{1+\varepsilon} D^{-1}.
\end{aligned} \tag{47}$$

By (45) – (47) it follows

$$S_c^{(2)}(X) \ll X^{1+\varepsilon} z^{-\frac{1}{2}}. \tag{48}$$

Using (1) and (48) we find

$$S_c^{(2)}(X) \ll X^{\frac{2c+1}{4}+\varepsilon}. \tag{49}$$

Bearing in mind (8), (43) and (49) we obtain

$$S_c(X) = \frac{1}{2}\sigma X + \mathcal{O}\left(X^{\frac{2c+1}{4}+\varepsilon}\right), \tag{50}$$

where σ is denoted by (4).

The Theorem is proved.

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